

Squeezed Spin States — Squeezing the Spin Uncertainty Relations —

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Abstract

The notion of squeezing in spin systems is clarified and principle for *spin squeezing* is shown. Two *twisting* schemes are proposed as building blocks for spin squeezing and are shown to reduce the standard quantum noise, $\frac{S}{2}$, of the coherent S -spin state down to the order of $S^{1/3}$ and $\frac{1}{2}$. Applications to partition noise suppression are briefly discussed.

1 Introduction

First, we will review the uncertainty relations and coherent states of spin [1] compared to those of boson. Then we will define squeezing in spin systems and show the principle for *spin squeezing* [2]. Secondly, we will propose fundamental schemes for spin squeezing, namely, *one-axis twisting* and *two-axis counter-twisting*, and discuss their limits [2]. Finally, applications are briefly discussed.

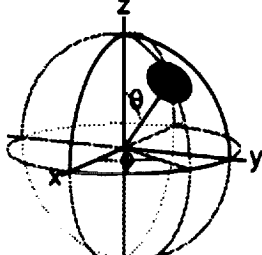
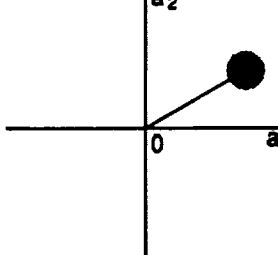
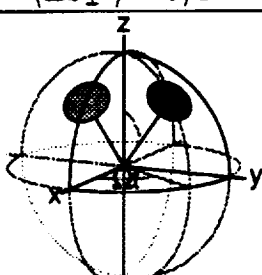
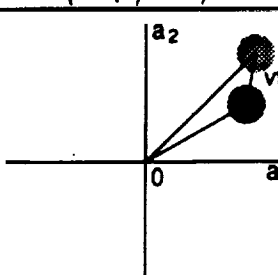
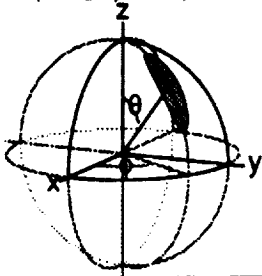
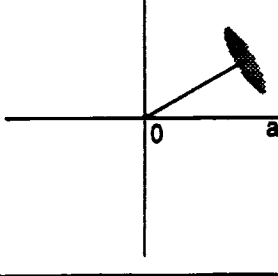
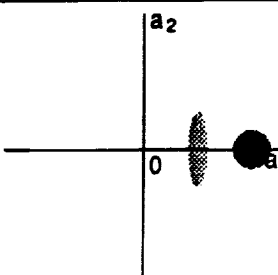
2 Uncertainty Relations — Spin vs. Boson —

Let us begin by comparing spins and bosons with respect to their uncertainty relations (TABLE I.) The spin commutation relation is $[S_i, S_j] = iS_k$, where $S_{i,j,k}$ are orthogonal spin components and the relation holds for any permutation of i, j, k . The same is true for associated uncertainty relations, $\langle \Delta S_i^2 \rangle \langle \Delta S_j^2 \rangle \geq |\langle S_k \rangle|^2 / 4$. This is quite different from the boson uncertainty relation since the right hand side (RHS) is state-dependent [3].

The coherent states can be defined as the minimum and equal uncertainty state; the state that minimizes the left hand side with the two uncertainties being equal. The eigenstate of the spin component of a certain direction (θ, ϕ) , $S_{\theta, \phi} = S_x \sin \theta \cos \phi + S_y \sin \theta \sin \phi + S_z \cos \theta$, with eigenvalue S satisfies this condition if S_k is the eigen component (which is S) and S_i and S_j are normal components (whose variances are $S/2$). This state is called a coherent spin state (CSS), Bloch state, or directed angular momentum state [1].

Before talking about squeezing, let's look at the linear motions. A linear Hamiltonian proportional to an arbitrary spin component rotates the spin vector about an axis. This is a precession.

TABLE I. Spin vs. boson with respect to uncertainty relation, coherent state and squeezing

	Spin	Boson
Uncertainty Relations	$[S_i, S_j] = iS_k$ \downarrow $\langle \Delta S_i^2 \rangle \langle \Delta S_j^2 \rangle \geq \langle S_k \rangle ^2 / 4$ $S_{\theta, \phi} \theta, \phi \rangle = S \theta, \phi \rangle$	$[a_i, a_j] = i/2$ \downarrow $\langle \Delta a_i^2 \rangle \langle \Delta a_j^2 \rangle \geq 1/16$ $a \alpha \rangle = \alpha \alpha \rangle$
Coherent States	 $\langle \Delta S_{\perp}^2 \rangle = S/2$	 $\langle \Delta a_i^2 \rangle = 1/4$
Translation	 $H = \hbar \Omega S_z$	 $H = \hbar i v a^\dagger + \text{H.c.}$
Squeezed States	$\times \quad \langle \Delta S_i^2 \rangle < \langle S_k \rangle /2$ $\langle \Delta S_{\perp}^2 \rangle < S/2$ 	$\langle \Delta a_i^2 \rangle < 1/4$ 
Squeezing	<p>Spin Squeezing ?</p> $\times \quad S_i(t) = g_i S_i(0), S_j(t) = g_j S_j(0)$ $H = ?$	 $a_1(t) = G^{-1} a_1(0), a_2(t) = G a_2(0)$ $H = \hbar i \chi (a^\dagger)^2 + \text{H.c.}$

It is regarded as a translation of the state on the spherical phase space of the spin. Although the rotation may change the uncertainties of the original spin components, the coherent spin state remains the minimum and equal uncertainty state as long as the component on the RHS is taken parallel to the mean spin vector.

Now let's discuss squeezing in a spin system. In a boson system, it is always regarded as squeezing if a certain quadrature amplitude has a variance smaller than the square root of the RHS of the uncertainty relation; that is $1/4$. If we define the squeezing of spin likewise [4] — a certain spin component has a variance smaller than the square root of the RHS — we can squeeze the spin by just rotating it. If this were really squeezing, the experimentalists would be very happy since they could do this easily. Unfortunately, it doesn't offer any improvement beyond the standard quantum limit.

The quantum limit of spin systems can be attributed to the directional uncertainties of the spin vector. Therefore the uncertainties normal to the mean spin vector are the relevant quantities to be squeezed. To eliminate the superficial coordinate dependency, we write the criterion of the spin squeezing as $\langle \Delta S_{\perp}^2 \rangle < S/2$ (one of the component normal to the mean spin vector has a variance smaller than $S/2$) [2].

The next problem is how to squeeze the spin. Boson squeezing is regarded as attenuation of one quadrature amplitude and amplification of the other by the same factor. This can be done by a degenerate parametric amplifier described by a quadratic nonlinear Hamiltonian. Geometrically it is an area-preserving linear transformation on the boson phase space \mathbf{R}^2 . The global shrinking/stretching is possible because boson phase space is an open plane.

In the spin case, permutative commutation relations obviously prohibit such a simple attenuation/amplification. In other words, global shrinking/stretching is impossible on the spherical phase space \mathbf{S}^2 of spin. The squeezing of spin is inevitably localized in phase space and, therefore, can be quite different from that of bosons.

3 Squeezed Spin States

Let's see how spin can, in principle, be squeezed. An S -spin system can be considered as a collection of a number, $2S$, of $1/2$ -spins. In the coherent spin state pointing up, all spins are up (Fig.1 (a)). Therefore the z -component of the total system is S . However, whether the x -component of each spin takes $1/2$ or $-1/2$ is completely independent and random. Therefore the variance is simply the sum of individual variances, $1/4$, which is $S/2$. The same is true for the y -component. These uncertainties are the origin of the standard quantum noise of the CSS. The spin vector \mathbf{S} is like a cone rather than an arrow. The diverging angle of the cone decreases with increasing S , since the base radius of the cone is proportional to \sqrt{S} .

In practical applications, it is desirable to reduce quantum noise for a given S . We have just seen that the origin of the standard quantum noise is a lack of quantum correlation among individual spins. If they are correlated, fluctuations of individual spins can cancel each other out (Fig.1 (b)). We refer to such a state as a squeezed spin state (SSS) [2]. Such a state can be conceived as an elliptical cone [5].

One way to establish the quantum correlations among individual spins is to let them interact with each other. This is a nonlinear interaction. Another way is to let them interact with an already correlated system such as squeezed light.

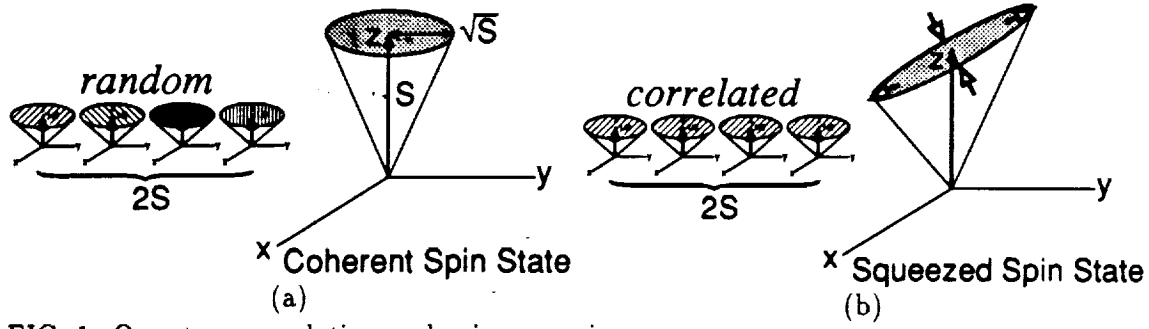


FIG. 1. Quantum correlation and spin squeezing

3.1 One-axis twisting

Let's consider the simplest nonlinear Hamiltonian, the square of a spin component, for example, $H = \hbar\chi S_z^2$. This interaction leads to $S_+(\mu) = S_+(0) \exp[i\mu(S_z + \frac{1}{2})]$, a rotation proportional to S_z , where μ is the strength of the interaction. If the initial state is on the equator of the sphere (Fig. 2 (a)), the interaction twists the noise distribution (Fig. 2 (b)).

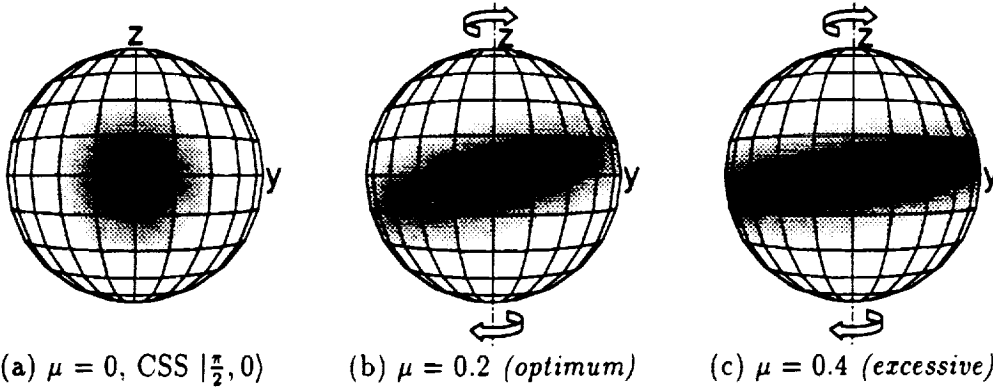


FIG. 2. Quasi-probability distribution $Q(\theta, \phi)$ [6] for one-axis twisting. ($S = 20$, $\mu = 2\chi t$).

The increased and decreased variances are,

$$V_+ \approx \frac{S}{2} (\mu S)^2$$

$$V_- \approx \frac{S}{2} \left[(\mu S)^{-2} + \frac{1}{24} (\mu^2 S)^2 \right] \geq 24^{-1/3} S^{1/3}$$

squeezing swirliness (at $\mu = 24^{1/6} S^{-2/3}$)

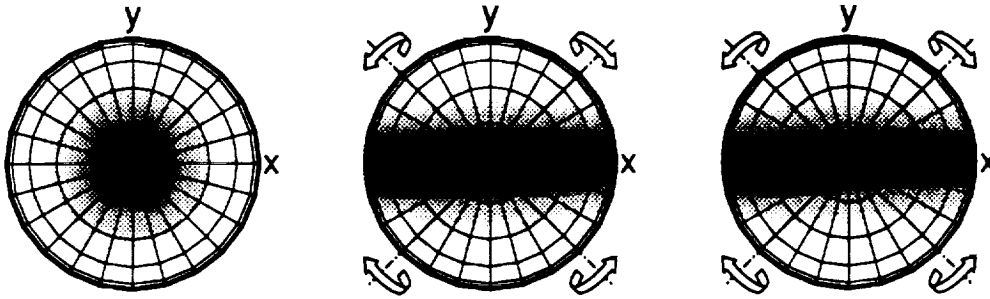
where $\mu S > 1$ and $\mu^2 S \ll 1$ are assumed. The noise distribution is stretched by a factor of μS in a certain direction, while it is shrunk by the same factor in the orthogonal direction. This is nothing but squeezing. However, the stretching of the distribution is not exactly along a geodesic of the sphere, it is slightly S-shaped. The second term arises from this non-ideal effect, swirliness. The deviation from the geodesic becomes comparable to the reduced width of the distribution when μ is increased to the order of $S^{-2/3}$. Then the variance reaches its minimum of $S^{1/3}$. Because of the swirliness, it is impossible to further reduce the quantum noise by means of one-axis twisting.

3.2 Two-axis counter-twisting

The swirliness can be canceled out if we twist the noise distribution simultaneously clockwise and counterclockwise about two orthogonal axes both normal to the mean spin vector (Fig. 3 (b)). This can be done, for example, by the following Hamiltonian,

$$H = \hbar\chi(S_{\frac{\pi}{2},\frac{\pi}{4}}^2 - S_{\frac{\pi}{2},-\frac{\pi}{4}}^2) = \frac{\hbar\chi}{2i}(S_+^2 - S_-^2)$$

We refer to this as two-axis counter-twisting. The noise distribution is shrunk along a geodesic and stretched along the orthogonal geodesic until it spans almost half the sphere. If we twist the distribution more, it splits into two and no further improvement occurs.



(a) $\mu = 0$, CSS $|0,0\rangle$ (b) $\mu = 0.203$ (*optimum*) (c) $\mu = 0.248$ (*excessive*)

FIG. 3. Quasi-probability distribution $Q(\theta, \phi)$ for two-axis counter-twisting. ($S = 20$, $\mu = 4\chi t$).

3.3 Limits of noise reduction

Let's compare the minimum variances of two kinds of squeezed spin states. The dots show the exact minimum attainable variances calculated numerically (Fig. 4). The variance of the ordinary coherent spin state increases linearly with S . One-axis twisting can reduce it to the order of $S^{1/3}$. Two-axis counter-twisting can further reduce it to $1/2$.

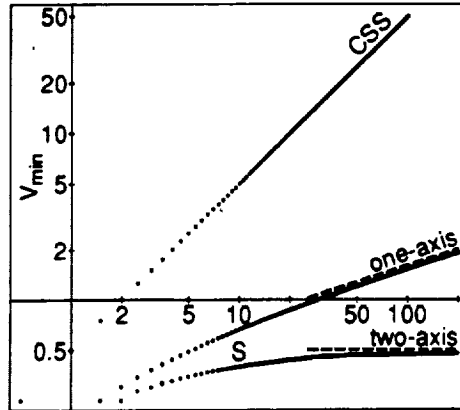


FIG. 4. Minimum variances vs. S

4 Applications to Partition Noise Suppression

There are many systems which can be described as a spin system. Spin squeezing described here offers better performance in these systems. For example, dispersion-less beamsplitters and interferometers for bosons and fermions can be described as a spin system with S being the

half of the total particle number N passing through them [7, 5]. The operator S_z corresponds to the half of the particle number difference $N_A - N_B$ between two paths (A and B), and S_+ transfers a particle from one path (A) to the other (B). The outputs of 50% beamsplitters (i.e., $\langle S_z \rangle = 0$) have the number and phase partition noises $\delta N \equiv [\Delta(N_A - N_B)]^2]^{1/2} = 2\langle \Delta S_z^2 \rangle^{1/2}$ and $\delta \phi \equiv [\Delta(\phi_A - \phi_B)]^2]^{1/2} \approx \langle \Delta S_y^2 \rangle^{1/2} / |\langle S_x \rangle|$. For ordinary linear beamsplitters, they are $\delta N = \sqrt{N}$ and $\delta \phi \approx 1/\sqrt{N}$ since the output is in CSS $|\pi/2, 0\rangle$. Their ratio can be changed by *spin squeezing* without violating the uncertainty principle $\delta N \delta \phi \geq 1$. Physically, they can be realized as nonlinear interferometers. Both self-phase-modulation $H_I = \hbar \chi (N_A^2 + N_B^2) = 2\hbar \chi (N^2/4 + S_z^2)$ of particles in both paths and mutual-phase-modulation $H_I = \hbar \chi N_A N_B = \hbar \chi (N^2/4 - S_z^2)$ between particles in different paths lead to one-axis twisting. Optical Kerr effect and Coulomb interaction give these number-dependent phase modulations. These nonlinear beamsplitters can achieve either $\delta N \approx N^{1/6}$ or $\delta \phi \approx N^{-5/6}$ [8].

5 Summary

In summary, We have clarified the notion of squeezing in a spin system. Spin is squeezed if one of the components normal to the mean spin vector has a variance smaller than $S/2$. We have shown the principle for spin squeezing. The spin can be squeezed by establishing quantum correlations among elementary spins. We have proposed the fundamental schemes for spin squeezing. One-axis twisting can reduce the noise down to $S^{1/3}$ and two-axis counter-twisting can reduce it to $1/2$. We have also discussed possible applications of spin squeezing to the sub-quantum-limit partition of quanta. Partition noise in either particle number or phase can be suppressed with a nonlinear beamsplitter which performs spin squeezing.

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